REFLECTION AND REFRACTION OF A PLANE PLASTIC WAVE AT THE INTERFACE BETWEEN TWO HALF-SPACES

(OTRAZHENIE PLOSKOI PLASTICHESKOI VOLNY I PRELOMLENIE EE NA GRANITSE DVUKH POLUPROSTRANSTV)

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The problem considered is that of a plane plastic wave which is incident normally on a plane boundary which separates two elastic-plastic half-spaces. The problem formulated in this way is spatially one-dimensional, and a uniaxial stress-strain curve suffices for the description of the phenomena. In this article, only assumptions of a general nature are made about the properties of the stress - strain curve. The reflected and refracted waves, the reflection coefficient and its relation to the stress-strain curve are studied.

Real media in which the propagation of large disturbances must be studied (soils, structural elements, etc.) are almost always inhomogeneous. This inhomogeneity may either be characterized by a continuous distribution or appear in form of more or less sharp interfaces. In the latter cases it is condidered that different media are in contact along some bounding surface. When a wave is incident on such a surface, reflection and refraction take place. For plastic waves the study of these phenomena is in its initial stage of development. It is natural that the study begin with the case of a plane wave incident normally on a plane interface between two media. A particular problem with a piecewise linear stress-strain curve is considered in [1].

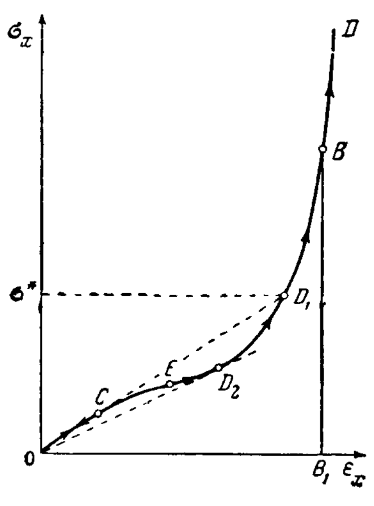

Fig. 1

It is assumed that the initial part of the (compression) stress-strain curve, corresponding to elastic straining, is a straight line (OC in Fig.1). On this segment loading and unloading take place along the same curve. In the remainder of the stress-strain curve for loading it is assumed only that the branch $CEBD$ is either everywhere concave upward or is divided by the point E into two parts: CE is convex upward and ED is concave. The curve $OCEBD$ represents a monotonously increasing function. As for unloading, it is assumed that to the right of the point C where the straining becomes inelastic, unloading occurs at constant density (on a straight line parallel to the axis of ordinates). Reloading is described by motion of a point upward along BB_1 up to the point B and, for further loading, along the branch BD. Each of the half-spaces in contact is described by a similar stress-strain curve; the two curves differ only quantitatively. As in [2], it is assumed that the incident wave is caused by a shock loading which decreases monotonously from its initial value (typical of the properties of a wave due to an explosion). The incident wave has a shock front which propagates into undisturbed medium, the particles behind the shock front undergoing unloading. This type of incident wave is possible in two cases:

1. The incident wave is elastic; a point on the segment OC corresponds to the stress at the shock front.

2. The incident wave is plastic; a point on the segment D_1D $(\sigma_x > \sigma^*)$ corresponds to the stress at the shock front. Points on a segment parallel to the axis of ordinates correspond to stresses at particles behind the front.

Under certain conditions the character of the incident wave which has s been described leads to similar reflected and refracted (transmitted) waves. Here the following cases are possible:

TABLE 1

	Incident wave	Reflected wave	Refracted wave
1	Plastic	Plastic	Plastic
2	"	"	Elastic
3	Elastic	"	Plastic
4	"	"	Elastic
5	"	Elastic	Plastic
6	"	"	Elastic

Case 6 is well known in the theory of elasticity. In this paper only cases 1 and 2 are considered. Cases 3, 4 and 5 may be examined similarly. The problem consists of giving quantitative descriptions of the incident, reflected and refracted waves (and, in particular, of finding coefficients of reflection and refraction) and also of the conditions for which the special cases mentioned above are realized.

1. In the following we shall denote the stress σ_x and the strain ϵ_x by σ and ϵ, respectively. Compressive stresses and compressive strains will be considered positive. Let the stress-strain curve for the half-space in which the incident wave propagates be specified by the relation

$$\sigma = \sigma^0 f(\varepsilon), \qquad \sigma^0 > 0$$

We shall examine the problem in the Lagrangian coordinates h, t, so that

$$x(h, t) = h + u(h, t)$$

where u is the displacement and x is the Eulerian coordinate. In the plane $h = 0$ an external loading $\sigma = \sigma_0(t)$ is given, with $\sigma_0(0) \equiv \sigma_0 \neq 0$; the function $\sigma_0(t)$ is monotonously decreasing. If σ_0 is sufficiently large, the particles behind the shock front will undergo plastic unloading at constant density. In this case we have Equations

$$\frac{\partial \sigma}{\partial h} + \rho_0 \frac{\partial v}{\partial t} = 0, \qquad \frac{\partial x}{\partial h} = \frac{\rho_0}{\rho(h)} \tag{1.1}$$

where ρ_0 is the initial density, ρ is the density of particles behind the front of the incident wave $(\rho > \rho_0)$. From (1.1) we obtain

$$x(h,t)=\int_0^h \frac{\rho_0 d\eta}{\rho(\eta)}+x_0(t), \qquad v(h,t)=\frac{\partial x}{\partial t}=x_0'(t) \tag{1.2}$$

$$\sigma(h,t)=-\rho_0 x_0''(t)h+\sigma_0(t) \tag{1.3}$$

At the shock front (starred quantities will refer to values at the shock front)

$$v_*=\varepsilon_* h_*', \qquad \sigma_*=\rho_0\varepsilon_* h_*'^2 \tag{1.4}$$

Eliminating v_* and σ_* from (1.3) and (1.4) after an integration, we obtain

$$\varepsilon_* h_* h_*'=\int_0^t \frac{\sigma_0(\tau)\,d\tau}{\rho_0}=F(t) \tag{1.5}$$

Comparing the second of Equations (1.4) with the stress-strain law, we have

$$\rho_0\varepsilon_* h_*'^2=\sigma^\circ f(\varepsilon_*), \qquad h_*'^2=\frac{\sigma^\circ}{\rho_0}\frac{f(\varepsilon_*)}{\varepsilon_*} \tag{1.6}$$

If the point representing the stress at the shock front lies on the segment $D_s D$ (Fig.1), then

$$f'(\varepsilon_*)\geqslant\frac{f(\varepsilon_*)}{\varepsilon_*}$$

This means that

$$\frac{d}{d\varepsilon}\frac{f(\varepsilon)}{\varepsilon}=\frac{\varepsilon f'(\varepsilon)-f(\varepsilon)}{\varepsilon^2}\geqslant 0$$

i.e. that the function $f(\varepsilon)/\varepsilon$ is monotonously increasing. Therefore, Equation (1.6) has a unique solution for ε_*, and there exists a monotonously increasing function $\varphi(\varepsilon_*)$

$$\varepsilon_*=\varphi(h_*')$$

which is the solution of Equation (1.6).

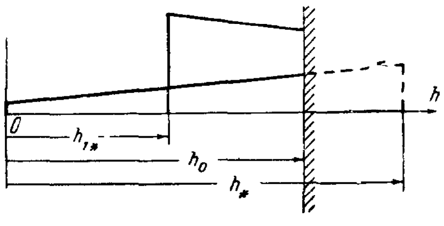

Fig. 2

As a result, we obtain a first-order differential equation to determine the Lagrangian coordinate $h_*(t)$ of the front

$$h_*'\varphi(h_*')h_*=\int_0^t \frac{\sigma_0(\tau)\,d\tau}{\rho_0} \tag{1.7}$$

which is to be solved with the initial condition $h_*(0)=0$.

2. Let us examine the reflected wave, assuming that it has a shock front with an associated stress that is increased as a result of the reflection. The condition for this to occur is that the second medium be "stiffer" than the first. The precise meaning of this requirement will be made clear later. We shall indicate quantities which refer to the reflected wave by the subscript 1. The reflected wave is constructed in the same way as in [2] in which a special form of the stress-strain law was assumed.

To avoid repetition we shall not enlarge on the details here, referring the reader to the reference cited.

The location of the front at some time after the reflection is shown in

Fig.2. Here h_0 denotes the distance from the initial plane to the interface, h_* is the distance to the front of the incident wave if this wave continued to propagate without encountering the interface. For the reflected wave, we have, just as for the incident wave,

$$\sigma_1(h, t) = -\rho_0 v_1'(t)(h - h_0) + \sigma_1(h_0, t)$$

$$x_1(h, t) = \int_0^h \frac{\rho_0 d\eta}{\rho(\eta)} + x_{10}(t), \quad v_1(h, t) = \frac{\partial x_1}{\partial t} = x_{10}'(t) \tag{2.1}$$

At the reflected shock front, (see [2]) (2.2)

$$v_{1*} - v = -[\varepsilon_1(h_{1*}) - \varepsilon(h_{1*})] h_{1*}', \qquad \sigma_b - \sigma_{1*} = -\rho_0 [v_{1*}(t) - v(t)] h_{1*}$$

where $\sigma_b(h)$ is the stress which existed at the particle with the coordinate h when the incident wave passed it.

For stress at the shock front, the stress-strain law has the form

$$\sigma_{1*} = \sigma^\circ f[\varepsilon_1(h_{1*})] \tag{2.3}$$

Eliminating ε_{1*} and σ_{1*} from Equations (2.2) and (2.3), we obtain the relation

$$\frac{1}{f[\varepsilon(h_{1*})]} f\{\varepsilon(h_{1*})[1 - \beta(\varepsilon(h_*) h_*' / \varepsilon(h_{1*}) h_{1*}')]\} = 1 - \beta \frac{f[\varepsilon(h_*)]}{f[\varepsilon(h_{1*})]} \frac{h_1^{*\prime}}{h_*'}$$

$$\left(\beta = 1 - \frac{v_{1*}(t)}{v(t)}\right) \tag{2.4}$$

Since the particle velocities in the reflected wave do not depend on the coordinate, they are the same at the shock front as at the interface separating the media. Denoting the velocity of the points of the interface by $V(t)$, we have $v_{1*}(t) = V(t)$. Equation (2.4) contains the unknown function $h_{1*}(t)$ and $V(t)$. This equation is a nonlinear, first-order differential equation in $h_{1*}(t)$ (the generalization of Equation (3.9) of [2]. Assuming that

$$|\delta| = \left|\beta \frac{h_*'}{h_{1*}'} \varepsilon(h_{1*})\right| \ll 1$$

which is prompted by physical considerations, we can replace Equation (2.4) by an approximate one found by expanding the function $f[\varepsilon(h_{1*}) - \delta]$ into a power series and retaining two terms. We then obtain Equation

$$\frac{dh_*}{(f[\varepsilon(h_*)]/\varepsilon(h_*))^{1/2}} = \frac{-dh_{1*}}{(f'[\varepsilon(h_{1*})])^{1/2}} \tag{2.5}$$

We note that in this approximation the dependence on $V(t)$ has dropped out of the equation, which can be explained by the comparatively small effect of the velocity of the points of the interface on the propagation of the shock front of the reflected wave. We shall investigate the case

$$\frac{f(\varepsilon)}{\varepsilon} \ll f'(\varepsilon), \qquad f'[\varepsilon(h_*)] < f'[\varepsilon(h_{1*})], \qquad h_*' < h_{1*}'$$

This means that the reflected wave front is propagated at a higher velocity than the (fictitious) incident wave front. In particular, this is also valid at the instant the reflection begins ($h_* = h_{1*} = h_0$). It should be remarked that we have denoted by "reflected wave" all motions which appear as a result of a reflection, not the part which is added on to the incident wave as is usual in the linear theory of elasticity and in acoustics.

3. In order to solve the problem completely, it is necessary to formulate a second equation containing the unknown function $V(t)$. To do this we must consider the wave which passes into the second medium (the refracted wave). We shall examine the case in which the refracted wave is elastic (case 2). We then have

$$u_2(h, t) = F(a_2t - h + h_0), \qquad \sigma_2(h, t) = a_2^2\rho_2 F'(a_2t - h + h_0)$$
$$v_2(h, t) = a_2 F'(a_2t - h + h_0) \tag{3.1}$$

On the plane of contact the conditions of continuity must be satisfied.

$$v_1(t) = v_2(h_0, t) = V(t), \qquad \sigma_1(h_0, t) = \sigma_2(h_0, t) \tag{3.2}$$

These two conditions and the second relation of (2.2) enable us to eliminate the function F and the stress σ_{1*}; we obtain the equation for $V(t)$

$$\rho(h_{1*} - h_0)V^{\cdot}(t) - (a_2\rho_2 - \rho_0 h_{1*}')V(t) = \rho_0 v h_{1*}' - \sigma_b \tag{3.3}$$

This is a linear, first-order differential equation. At the instant of reflection the coefficient $\rho_0(h_{1*} - h_0)$ of the derivative goes to zero. This causes the general solution of the homogeneous equation to be unbounded. For if we take the instant when the wave strikes the interface as $t = 0$, then we have for small $t > 0$ $\quad h_{1*} - h_0 = -a_1t + \ldots$

$$-\rho_0 a_1 t V^{\cdot}(t) - (a_2\rho_2 + a_1\rho_0)V(t) = -\rho_0 v(0)a_1 - \sigma_b + \ldots$$

We shall seek a solution in the form

$$V(t) = Ct^{\alpha} + V_0 + \ldots \tag{3.4}$$

where C, V_0 and α are constants and the first term on the right-hand side is the general solution of the homogeneous equation. We obtain the values

$$\alpha = -1 - \frac{a_2\rho_2}{a_1\rho_0}, \qquad V_0 = \frac{\rho_0 v(0)a_1 + \sigma_b}{a_2\rho_2 + a_1\rho_0} \tag{3.5}$$

for α and V_0.

Since $\alpha < 0$, the general solution is unbounded as $t \to 0$, and the condition of boundedness of the solution as $t \to 0$ must be imposed as the initial condition in the solution of (3.3). The constant V_0 is the initial value of the velocity of the points of the interface, which is expressed in terms of the as yet unknown initial velocity a_1 of the reflected wave. (If the wave is incident on an obstacle, the initial value of velocity of the obstacle is equal to zero).

4. Let us investigate in greater detail the special case in which the incident wave has the form of a step. This case is all the more interesting because it describes asymptotically for the initial stages the phenomenon of reflection for an incident shock wave a general form. If the incident wave has the form of a step the analysis of the situation is simplified consider-

ably since the parameters of the wave behind the front are constant. Thus, in the incident wave, behind the front

$$\varepsilon(h) = \varepsilon_0, \quad v(t) = v_0, \quad \sigma_b = \sigma_0(0) = \rho_0 v_0 a, \quad a > 0, \quad h_*'(t) = a$$

Let us try to satisfy all the conditions of the problem by assuming that the parameters are also constant in both the reflected and refracted waves

$$\varepsilon_1(h) = \varepsilon_1, \quad v_1(t) = v_1, \quad h_{1*}'(t) = -a_1, \quad a_1 > 0$$

$$h_{1*} - h_0 = -a_1 t$$

$$\varepsilon_2(h, t) = \varepsilon_2, \quad v_2(h, t) = v_2, \quad h_{2*}'(t) = a_2, \quad a_2 > 0$$

Equation (2.4) then assumes the form

$$f\left[\varepsilon_0\left(1 + \beta\frac{a}{a_1}\right)\right] = f(\varepsilon_0)\left\{1 + \beta\frac{a_1}{a}\right\}, \qquad \beta = 1 - \frac{V}{v} = \mathrm{const} \tag{4.1}$$

In order to solve this equation approximately, we set $\beta a/a_1 = \sigma$.

We may then rewrite Equation (4.1) as

$$\left(\frac{a_1}{a}\right)^2 = \frac{f(\varepsilon_0 + \varepsilon_0\sigma) - f(\varepsilon_0)}{\sigma f(\varepsilon_0)}$$

The expansion of the right-hand side in powers of $\sigma\varepsilon_0$ permits us to represent the relation between a_1/a and β in the following parametric form:

$$\frac{a_1}{a} = \left\{\frac{f'(\varepsilon_0)\varepsilon_0}{f(\varepsilon_0)} + \frac{f''(\varepsilon_0)\varepsilon_0^2}{2f(\varepsilon_0)}\sigma + \cdots\right\}^{1/2}, \qquad \beta = \sigma\left\{\frac{f'(\varepsilon_0)\varepsilon_0}{f(\varepsilon_0)} + \frac{f''(\varepsilon_0)\varepsilon_0^2}{2f(\varepsilon_0)}\sigma + \cdots\right\}^{1/2} \tag{4.2}$$

In Equations (4.2) the quantity σ plays the rôle of a parameter. The retention of just the first term of the series yields the same result as Equation (2.5), namely

$$\frac{a_1}{a} = \left(\frac{f'(\varepsilon_0)}{f(\varepsilon_0)/\varepsilon_0}\right)^{1/2} \tag{4.3}$$

We now return to Equation (3.4). For a bounded solution in the form of a constant we obtain

$$V_0 = \frac{\sigma_b + \rho_0 v_0 a_1}{a_2\rho_2 + a_1\rho_0} = \frac{v_0\rho_0(a + a_1)}{a_2\rho_2 + \rho_0 a_1} \tag{4.4}$$

This value of the interface velocity allows us to compute the stress at interface in the reflected wave

$$\sigma_2(h_0, t) = \rho_2 a_2 v_2 = \frac{\rho_0 v_0 a(1 + a_1/a)}{1 + \rho_0 a_1/\rho_2 a_2} \tag{4.5}$$

According to condition (3.2), the stress in the refracted wave has the same value.

It is useful to introduce the concept of the reflection coefficient (and the refraction coefficient), where we mean by this the ratio of the stress in the reflected (refracted) wave to the stress in the incident wave. For a step wave this ratio does not depend on time; for an arbitrary wave it will be referred to the instant of incidence ($t = 0$) and should be computed by the same formula as for the step wave (assuming that the incident wave is a shock wave). We denote the reflection (refraction) coefficient by K .

Then

$$K = \frac{\sigma_1(h_0, t)}{\sigma_b} = \frac{1 + a_1/a}{1 + \rho_0 a_1/\rho_2 a} \qquad (4.6)$$

This equation shows that the reflection coefficient depends significantly on the ratio a_1/a. Likewise, K depends on the ratio a_2/a. In particular, if the second medium is infinitely stiff ($a_2 = \infty$), the phenomenon reduces to the reflection from an immovable wall and the reflection coefficient in this case assumes the value

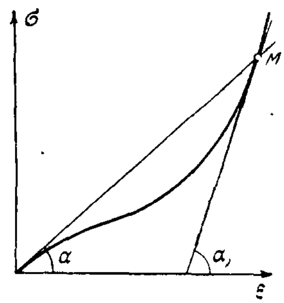

Fig. 3

$$K = 1 + a_1/a \qquad (4.7)$$

The ratio a_1/a is determined from the stress-strain curve according to Equation (4.3) and can be interpreted geometrically. As a matter of fact, this ratio can be expressed as

$$\frac{a_1}{a} = \frac{f'(\varepsilon_0)}{f(\varepsilon_0)/\varepsilon_0} = \frac{\tan\alpha_1}{\tan\alpha}$$

where the angles α and α_1 are shown in Fig.3.

The dependence of the reflection coefficient on the intensity of the incident wave is also determined by the stress-strain law. Ler us give some examples.

1. Let the part of the stress-strain curve under consideration be expressed analytically by the power relation $f(\varepsilon) = \varepsilon^n$. Then

$$a_1/a = \sqrt{n}, \qquad K = 1 + \sqrt{n}$$

In this case the reflection coefficient does not depend on the intensity of the incident wave, as was noted in [2].

2. A different result is obtained for the case when for $A/B < \varepsilon$

$$f(\varepsilon) = B\varepsilon - A, \quad B > 0, \quad A > 0, \quad K = 1 + \left(\frac{B}{B - A/\varepsilon}\right)^{1/2}$$

As we see, the reflection coefficient decreases with increasing intensity of the incident wave.

3. Finally, let

$$f(\varepsilon) = \frac{m}{(\varepsilon^\circ - \varepsilon)^p}, \quad 0 < \varepsilon < \varepsilon^\circ, \quad m > 0, \quad p > 0$$

Then

$$K = 1 + \left(\frac{p}{\varepsilon^\circ/\varepsilon - 1}\right)^{1/2}$$

and the reflection coefficient increases with increasing intensity of the incident wave.

In conclusion, the conditions for the realization of the case of reflection which has been considered should be noted. Besides the condition that the incident disturbance be a shock wave, it is also necessary that the refracted wave in the elastic range, i.e. that the stress in the second medium at the interface

$$\rho_0 v_0 a\, \frac{1 + a_1/a}{1 + \rho_0 a_1/\rho_2 a_2}$$

be less than the corresponding elastic limit, and that the reflected wave be stronger than the incident, i.e. $K > 1$. This last condition leads to the inequality $a_2\rho_2 > a\rho_0$, which means that the acoustic impedence of the elastic medium must be greater than the "effective" acoustic impedence of the first medium.

5. We now proceed to the case in which the refracted wave is plastic (case 1). Equation (2.4) remains valid in this case because the properties of the second medium enter into it only through the value of the velocity $V(t)$ of the points of the interface, and this velocity occurs in Equation (2.4) as an unknown quantity. In contrast to the preceding, Equation (3.3) does undergo a change.

For the refracted wave, just as for the incident wave in Section 1, we obtain

$$\sigma_2 (h, t) = - \rho_2 v_2^{\bullet} (h - h_0) + \sigma_2 (h_0, t) \tag{5.1}$$

For the reflected wave, we have

$$\sigma_1 (h, t) = - \rho_0 v_1^{\bullet} (h - h_0) + \sigma_1 (h_0, t) \tag{5.2}$$

We then obtain the values of these stresses at the reflected and refracted shock fronts, respectively

$$\sigma_2 (h_{2*}, t) = - \rho_2 v_2 \ (h_{2*} - h_0) + \sigma_2 (h_0, t) = \rho_2 v_{2*} h_{2*} \tag{5.3}$$

$$\sigma_1(h_{1*},t) = - \rho_1 v_1^{\cdot}(h_{1*} - h_0) + \sigma_1(h_0,t) = \rho_0 v_{1*} h_{1*}' + \sigma_b - \rho_0 v h_{1*} \tag{5.4}$$

On the intreface between the media ($h = h_0$), the boundary conditions

$$v_1 (h_0, t) = v_2 (h_0, t), \qquad \sigma_1 (h_0, t) = \sigma_2 (h_0, t) \tag{5.5}$$

must hold.

Since the functions v_1 and v_2 do not depend upon h, the first condition of (5 5) becomes the relation

$$v_1 (t) = v_2 (t) = V (t) \tag{5.6}$$

where $V(t)$ is the velocity of the particles of the interface.

Taking (5.6) and (5.5) into account, and subtracting (5.4) from (5.3), we obtain the following differential equation for $V(t)$:

$$[\rho_2 (h_{2*} - h_0) - \rho_0 (h_{1*} - h_0)] V^{\cdot} (t) + (\rho_2 h_{2*}' - \rho_0 h_{1*}') V (t) = \sigma_b - \rho_0 v h_{1*}' \tag{5.7}$$

This equation appears in the new problem in place of Equation (3.3).

For small positive values of time, we have

$$h_{1*} - h_0 = - a_1 t + \ldots, \qquad h_{2*} - h_0 = a_2 t + \ldots$$

and, under this condition, Equation (5.7) may be rewritten as

$$(\rho_2 a_2 + \rho_0 a_1)\, t V^{\cdot} (t) + (\rho_2 a_2 + \rho_0 a_1) V (t) = \sigma_b - \rho_0 v h_{1*}$$

The corresonding homogeneous equation has the general solution $V(t) = ct^{-1}$. The requirement of boundedness of the solution of Equation (5.7) in the vicinity of $t = 0$ then forces us to set $c = 0$.

In order to solve the problem one more equation must be formulated. At the front of the refracted wave which propagates, as assumed, into an undisturbed medium, the conditions

$$v_{2*} = \varepsilon_{2*} h_{2*}', \qquad \sigma_{2*} = \rho_2 \varepsilon_{2*} h_{2*}'^2$$

hold . From this we find

$$\varepsilon_{2*} = \frac{\sigma_{2*}}{\rho_2 h_{2*}'^2}$$

or, taking account of the stress-strain relation in the second medium $\sigma_2 = \sigma_2^\circ f_2(\varepsilon_2)$, we finally obtain

$$h_{2*}'^2 = \frac{\sigma_2^\circ}{\rho_2} \frac{f_2(\varepsilon_{2*})}{\varepsilon_{2*}} \tag{5.8}$$

Replacing ε_{2*} in accordance with the relation

$$\varepsilon_{2*} = \frac{v_{2*}}{h_{2*}'} = \frac{V(t)}{h_{2*}'}$$

we obtain Equation

$$\frac{f_2(V / h_{2*}')}{V / h_{2*}'} = \frac{\rho_2 h_{2*}'^2}{\sigma_2^\circ} \tag{5.9}$$

If the function which is the inverse of $\varepsilon^{-1}[f'(\varepsilon)]^2 = \eta$, is denoted by $\varepsilon = \varphi(\eta)$, Equation (5.9) can be replaced by the relation

$$\frac{V}{h_{2*}'} = \varphi_2 \left(\frac{\rho_2}{\sigma_2^\circ} h_{2*}'^2\right) \tag{5.10}$$

The problem formulated at the beginning of Section 5 then reduces to the solution of three simultaneous equations (2.4), (5.7) and (5.9) or (5.10). These equations contain unknown functions of time h_{1*}, h_{2*} and V . For a general stress-strain law it is necessary to resort to numerical solution preceded by a qualitative investigation of the equations. Equations (2.4) can, of course, be replaced here also by the approximation of Equation (2.5).

We shall dwell in some detail on the special case in which the incident wave has the form of a step. In this case the reflected and refracted waves will also have that same shape. Retaining the notation of Section 4, we obtain three equations to find the three unknown constants a_1, a_2, and V_0

$$f\left[\varepsilon_0\left(1 + \beta\frac{a}{a_1}\right)\right] = f(\varepsilon_0)\left(1 + \beta\frac{a_1}{a}\right), \quad V_0 = \frac{\rho_0 v_0 (a + a_1)}{a_1\rho_0 + a_2\rho_2}, \quad \frac{f_2(V_0 / a_2)}{V_0 / a_2} = \frac{\rho_2 a_2^2}{\sigma_2^\circ} \tag{5.11}$$

where

$$\beta = 1 - \frac{V_0}{v_0}, \qquad a = \left(\frac{\sigma^\circ}{\rho_0}\right)^{1/2}\left(\frac{f(\varepsilon_0)}{\varepsilon_0}\right)^{1/2}, \qquad v_0 = \left(\frac{\sigma^\circ}{\rho_0}\right)^{1/2}\sqrt{\varepsilon_0 f(\varepsilon_0)} \tag{5.12}$$

The first of Equations (5.11) may be replaced by the simpler approximate one, Equation (4.3). We obtain the previous Formula (4.6) for the reflection coefficient, but now the constant a_2 is not given beforehand. In order to determine this constant we use the second of Equations (5.11). We then obtain the relation

$$\frac{\rho_0 v_0 (a + a_1)}{a_2(\rho_0 a_1 + \rho_2 a_2)} = \varphi_2\left(\frac{\rho_2}{\sigma_2^\circ} a_2^2\right) \tag{5.13}$$

the left-hand side of which decreases monotonously with a_2, while the right-hand side increases monotonously. Therefore, Equation (5.13) has a unique solution which determines the velocity of the refracted wave front.

We shall now explain how the plastic properties of the medium affect the reflection. In order to do this we shall compare a reflection from an ideally elastic medium, in which the velocity of longitudinal waves is $(\sigma_2^\circ / \rho_2)^{1/2}$, with a reflection from a corresponding plastic medium in which the phenomena proceed as prescribed under case 1 in Table 1. Here the stresses in the shock wave are described by a portion of the stress-strain curve for which

$$\frac{f_2(\varepsilon)}{\varepsilon} = 1 + \psi(\varepsilon)$$

where $\psi(\varepsilon)$ is a positive function. Therefore,

$$\frac{f_2(V/h_{2*}')}{V/h_{2*}'} = 1 + \psi\left(\frac{V}{h_{2*}'}\right) = \frac{\rho_2}{\sigma_2^{\circ}} h_{2*}'^2, \qquad h_{2*}' > \left(\frac{\sigma_2^{\circ}}{\rho_2}\right)^{1/2}$$

Equation (4.6) shows that in passing from an ideally elastic medium to a plastic one in which reflection occurs as in case 1, the reflection coefficient increases. Finally it is necessary to show that case 1 can be realized. It does apply if the following three inequalities hold:

$$\sigma_* > \sigma^*, \qquad \sigma_{2*} > \sigma_2^*, \qquad K > 1$$

where σ^* is the stress corresponding to the point D_1 (Fig.1), $\sigma_2{}^*$ is the corresponding stress for the second medium. Written out in detail for step waves (and this is sufficient), these inequalities have the form

$$\rho_0 v_0 a > \sigma^*, \qquad \rho_2 V_0 a_2 = \rho_0 v_0 a K > \sigma_2^*, \qquad \rho_0 a < \rho_2 a_2$$

The first inequality requires that the incident wave be strong enough, which may always be assumed. The second inequality, when taken together with the third, reduces to the same thing. It remains to show that the third inequality can always be satisfied for any strong incident wave. In proving this statement, we shall limit ourselves to the case in which the curve of the function $f_2(\varepsilon)$ has a vertical asymptote $\varepsilon = \varepsilon^*$ and that $\rho_2 > \rho_0$. Then the curve of the function $\varepsilon = \varphi_2(\eta)$ has the horizontal asymptote $\varepsilon = \varepsilon^*$, with $0 < \varphi_2 < \varepsilon^*$.

Equation (5.13) can be rewritten as

$$\frac{\rho_0 a + \rho_0 a_1}{\rho_2 a_2 + \rho_0 a_1} \frac{\rho_0 a}{\rho_2 a_2} \frac{\rho_2}{\rho_0} = \frac{1}{\varepsilon_0} \varphi_2\left(\frac{\rho_2}{\sigma_2^{\,0}} a_2^2\right) \tag{5.14}$$

Let us assume that there is a sufficiently strong incident wave for which $\rho_0 a \geqslant \rho_2 a_2$. Then the left-hand side of the equation turns out to be greater than unity, while the right-hand side can be approximated by the number $\varepsilon^*/\varepsilon_0$ and can be made smaller than unity.

These considerations show that case 1 is definitely the actual one for a sufficiently intense incident wave.

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